

M. Memo with the Father's Compliment

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ON THE
APPLICATION OF ANALYSIS
TO THE
DISCOVERY OF LOCAL THEOREMS AND PORISMS.

BY
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ON THE
APPLICATION OF ANALYSIS
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THOSE who have devoted much of their attention to mathematical inquiries, must have had frequent occasion to admire the unexpected and intimate connection which subsists between branches of their science apparently the most remote and unconnected with each other; and repeated observation will have convinced them, that no researches, however recondite or abstruse, should be neglected, because they appear to stand isolated and detached from the body of the science. These reflections appear to have been felt with strongest force, by those who have most contributed to its advancement, and were particularly insisted on by MACLAURIN, in the Preface to his *Geometria Organica*. The present Paper will add another example, to instances already numerous, of the latent affinity between departments of mathematics, usually regarded as the most opposite.

By those who have studied the writings of the illustrious restorers of the most valuable and interesting portion of the ancient

cient geometry, the views which are here opened may probably be regarded with surprise, not unmixed with regret, that the increasing perfection of the language of symbols should gradually cause it to usurp the hitherto exclusive domains of the higher geometry. But whatever may be the discoveries to which the geometrical interpretation of language, at once the most comprehensive and condensed which human ingenuity has devised, shall give birth, the restorer of the Porisms of EUCLID, and the author of the General Theorems, will retain an undiminished reputation, and their works continue to be studied, by all those who wish to acquire a correct taste for the geometry of the ancients. Those propositions which have received the appellation of Local Theorems and Porisms, may, in one point of view, be considered as differing from theorems and problems, by having something more general, or indeterminate, in their nature; by affirming that some property is possessed not merely by one individual, but by every one, of some class or species. It is on this circumstance that the algebraic investigation of porisms is founded: the arbitrary constants of an equation of two variables, are made to submit to certain conditions, which shall leave the variables themselves still indeterminate. By generalising this process, we take, instead of arbitrary constants, unknown functions of the variables themselves; and instead of the resulting algebraic equations, we find functional equations which determine the form of the functions we have assumed.

This process, which will be better understood by the subsequent inquiries, leads us at once to the highest pitch of generality, and puts us in possession of innumerable porisms, and local theorems, each comprehending whole classes of curves. By limiting and determining the form of the arbitrary functions involved in the solution, we gradually restrict the extent

of our results, and we frequently arrive at *loci* belonging to the conic sections. Sometimes, however, the simplification is still greater, and we are surprised by recognising, amongst extensive classes of curves, possessed of some peculiar property, some well known *locus* of the straight line or circle.

The inquiries which follow are given nearly in the order in which they occurred; a different arrangement might have appeared more systematic, but it would have had the great disadvantage of concealing the means by which the results were arrived at. Several of the more restricted porisms and local theorems, might have admitted of a geometrical dress; but this would have been inconsistent with the object I had proposed to myself in the present paper. The greater part are, I believe, beyond the powers of geometry; and this opinion, if allowed to be correct, will perhaps, by the admirers of the ancient geometry, be admitted as some excuse for the present attempt to add to its stores, by means so very foreign.

There exist numerous classes of curves possessed of the following property:

If we take any abscissa AE, Plate XXII. Fig. 1, and ordinate BE, and if we make AF = BE, and find the new ordinate CF, and repeat the same process n times, the n th ordinate HI shall equal the first abscissa AE. It is easy to perceive, that if $y = \psi x$ represent the equation of the curve, then the equation determining the form of ψ will be

$$\psi^n x = x$$

As the classes of curves here alluded to, will frequently occur in the following inquiries, I shall venture to bestow on them the name of *Periodic Curves*, which was suggested by the similar name assigned by Mr HERSCHEL to the function which satisfies the equation just given. It will also be convenient to apply

ply some name to the series of ordinates thus found: I shall therefore call them *Corresponding Ordinates*. These curves may be divided into orders, according to the value of the index n ; if $n \pm 1$, the first order consists of only one curve, namely, $y = \psi x = x$, which is the equation of a right line, making an angle of 45° with the axis.

When $n = 2$, and ψ is determined from $\psi \cdot x = x$,

$$y = \psi x$$

represents all periodic curves of the second order, and so on.

The equation $\psi^n x = x$ has been solved by me in the *Philosophical Transactions* for 1816, and numerous examples are given in another paper, published in the following year. This solution has, however, been much improved by Mr HORNER (see *Annals of Philosophy*, October 1817), who has shewn that the algebraic equation to which my method of solution leads in all cases, admits of a ready solution. The form which is thus assigned to the function ψ is

$$\psi x = \varphi^{-1} \left\{ \frac{a + b x}{c - \frac{b^2 - 2 b c \cos \frac{k\pi}{n} + c^2}{2 (1 + \cos \frac{k\pi}{n}) a} x} \right\}$$

Amongst those included in the second order of periodic curves, will be found the right-angled hyperbola referred to co-ordinates parallel to its assymptotes, and also circles of all orders, where equations are $x^n + y^n = a^n$; these curves possess many singular properties. All those in which n is even are re-entering curves, without any infinite branches, whose form is nearly that of the subjoined figure, Fig. 2. In this curve, if we take any abscissa, and ordinate CD and BD, and if we turn the triangle CBD, into such a position that B shall coincide with the axis CF, then the point C will coincide with G some point

point in the curve, and also the arc GF, together with the arc BF, will be equal to the quadrant AF of the curve.

Let ABC be any periodic curve of the second order, whose equation is $y = \alpha x$, and DB, EC any two corresponding ordinates, it is required to find another curve AFG, such that the sum of its two ordinates, at the points D and E, may be always a constant quantity.

From the nature of the curve ABC, we have $AD = x$, and $AE = DB = \alpha x$; also $\alpha^2 x = x$.

Let $y = \psi x$ be the equation of the curve AFG, then $FD = \psi x$, and $GE = \psi(AE) = \psi \alpha x$, and by the prescribed condition, we have for the equation determining ψ

$$\psi x + \psi \alpha x = c;$$

the general solution of which is,

$$\psi x = \frac{c \phi x}{\phi x + \phi \alpha x};$$

the function ϕ is perfectly arbitrary, and of the class of curves comprehended in the equation $y = \frac{c \phi x}{\phi x + \phi \alpha x}$, we may enunciate the following porisms.

Any of this family of curves being given, a periodic curve of the second order (ABC), may always be found such, that if we take any two abscissæ in the curve given, respectively equal to any two corresponding ordinates of the curve found, and draw ordinates to the given curve, and if we prolong either of these ordinates (EG) above the curve, until the part above (GH) is equal to the first of the two ordinates, the extremity of the ordinate, thus increased, will always be situated in a right line given by position, Fig. 3.

The line thus given by position is parallel to the abscissæ, and situated at the distance denoted by c .

If we prolong either of the two ordinates (GE) below the abscissa, by a quantity (EL) equal to the first ordinate, the extremity of the ordinate, thus prolonged, will always be situated in a curve, similar and equal to the given curve, which is also given by position.

The position of the given curve is parallel to the original curve. The deduction of these porisms is so obvious, from the property of the class of curves, that I consider any farther explanation of them unnecessary. We shall, however, by assigning particular values to some of the functions, find some very simple results. Let us suppose $\alpha x = \sqrt{a^2 - x^2}$, then the family of curves will be contained in the equation,

$$y = \frac{c \varphi(x)}{\varphi x + \varphi(\sqrt{a^2 - x^2})}.$$

Of these curves, the following property may be stated :

Any curve of this family being given, a straight line (HK) may be found through any point, of which, if a line (HD) be drawn at right angles to the axis of the abscissa, it will cut off an abscissa (AD), and part of it will form the ordinate (CD), if an abscissa (AE) be found, whose ordinate (BE) is equal to that part of the line drawn, which is intercepted between the line found and the curve; and if both the two ordinates thus found, be prolonged below the axis, until the part of each below is equal to the abscissa belonging to the other ordinate, the two points to which these lines are prolonged, are situated in the circumference of a circle given by position. See Fig. 4.

Let $\alpha x = -x$, then the curves included in the family

$$y = \frac{c \varphi x}{\varphi x + \varphi(-x)}$$

possess the following property :

Any

Any curve of this kind being given, if we take two equal abscissæ, on opposite sides of the centre, and prolong the ordinates at both these points, until the parts of each produced be equal to the ordinate at the other point, then the locus of the points thus found is a right line given by position.

If we make $\phi x = a + x$, the curve becomes a right line, whose equation is $y = \frac{c}{2} + \frac{c}{a}x$; and the line found is parallel to the axis, and the curve given, is a right line passing through a point in the axis of the ordinates, equally distant from each parallel.

Let ABC be any periodic curve, Fig. 5. and DB, EC, any two corresponding ordinates; draw the ordinates FD, GC, and draw the line HFG, passing through the points F and G, it is required to find the equation of this line.

w and v being the co-ordinates of any point in this line, its equation will be $v = Aw + B$; the quantities A and B being determined by the condition, that it must pass through the points F and G, whose co-ordinates are respectively x and ψx , and αx , and $\psi \alpha x$, we have

$$v = \frac{\psi x - \psi \alpha x}{x - \alpha x} w + \frac{\alpha x \cdot \psi x - x \cdot \psi \alpha x}{x - \alpha x}.$$

Let us now determine the form of the curve AFG by this condition, that the line HG shall always be parallel to itself. It is well known, that the co-efficient of the abscissa w expresses the tangent of the angle of the inclination of the line to its axis; this must therefore be constant, and we have

$$\frac{\psi x - \psi \alpha x}{x - \alpha x} = c, \text{ or } \psi x - \psi \alpha x = c(x - \alpha x);$$

the solution of this equation is $\psi x = \frac{c(x - \alpha x)\phi x}{\phi x + \phi \alpha x}$.

This suggests the following porisms:

Any

Any of the family of curves contained in the equation $y = \frac{c(x - \alpha x) \phi x}{\phi x + \psi \alpha x}$ being given, a right line may be found, through any point of which, if a right line be drawn perpendicularly, cutting the curve in two points, and if the ordinate at each of these points is produced to a point below the axis, until the part below each is equal to the abscissa belonging to the other ordinate, the two points thus found will always be situated in a periodic curve, given in species and position.

The line to which the other lines are drawn perpendicular, is at right angles to the line HFG in the figure.

If we make $\alpha x = \sqrt{a^2 - x^2}$, the curve which is the locus of these points is a circle, whose radius is equal to a , and if we also make $\phi x = b$, the original curve is an ellipse.

If we make $\alpha x = \frac{a^2}{x}$, and also $\phi x = b$, the equation of the given curve is $y = \frac{c}{2} (x - \frac{a^2}{x})$, which is an hyperbola, and that of the curve found is $y = \frac{a^2}{x}$, a right-angle hyperbola.

If in the equation of the right line,

$$v = \frac{\psi x - \psi \alpha x}{x - \alpha x} w + \frac{\alpha x \cdot \psi x - x \cdot \psi \alpha x}{x - \alpha x},$$

we make $w = 0$, we have the distance of the point K from A; let us suppose this to remain constant, then

$$\frac{\alpha x \cdot \psi x - x \cdot \psi \alpha x}{x - \alpha x} = c, \text{ or } \alpha x \cdot \psi x - x \cdot \psi \alpha x = c(x - \alpha x);$$

the solution of which is $\psi x = \frac{1}{\alpha x} \cdot \frac{c(x - \alpha x) \phi x}{\phi x + \psi \alpha x}$.

The

The class of curves comprehended in the equation $\psi x = \frac{1}{\alpha x} \cdot \frac{c(x - \alpha x) \varphi x}{\varphi x + \varphi \alpha x}$, are possessed of the following property :

Any curve of the kind being given, a point (K) may be found, through which, if a right line (KG) be drawn in any direction, cutting the curve in two points (F and G), if the ordinate at one of these points be prolonged to a point below the axis, until the part below is equal in length to the abscissa corresponding to the other point of intersection, then the point thus found is always situated in a periodic curve of the second order, given in species and position. Fig. 5.

If in the equation of the line, we make $x = 0$, we have $w = \frac{\alpha x \cdot \psi x - x \cdot \psi \alpha x}{\psi x - \psi \alpha x}$, and if we suppose this to remain constant, we in fact fix the point K in the axis of the abscissæ. This gives the equation for determining ψ ,

$$(\alpha x - c) \psi x = (x - c) \psi \alpha x;$$

whose solution is $\psi x = \frac{c \mathcal{X} \left\{ \overline{x}, \overline{\alpha x} \right\}}{\alpha x - x}$.

Let AFG be any periodic curve, Fig. 6., and AD, AE any two corresponding abscissæ. Required the equation of all curves, such that the line joining the summits of the two ordinates, raised at the points D and E, shall be constant.

The equation of the periodic curve being $y = \alpha x$, let that of the family of curves sought be $y = \psi x$, then we have $AD = x$, $AE = \alpha x$, $BD = \psi x$, $CE = \psi \alpha x$, and the equation determining the form of ψ is

$$(\psi \alpha x - \psi x)^2 + (\alpha x - x)^2 = c^2. \quad (a)$$

c

This

This equation is one of those which I noticed in a paper in the *Philosophical Transactions* for 1817, p. 211. I there stated, that it appeared impossible for any function to satisfy the equation, unless it contained a radicle, and unless different roots were taken in different parts of the equation. This explanation will perhaps be rendered more satisfactory by the application to geometry. The general solution of the equation in question, is

$$\psi r = \pm \frac{\varphi x \cdot \sqrt{c^2 - (ax - x)^2}}{\varphi x + \varphi ax} = y,$$

where the upper sign must be used in one part of the equation, and the lower sign in another. From this equation we learn, that for every value of x there are two values of y , equal, but extending in opposite directions; or that the curve is symmetrical with regard to the axis of the x 's: and since we must use different values of the radicle, it appears that the two points B and C cannot be in the same branch of the curve, and on the same side of the axis, but that one point being situated above the axis, the other must be placed in the corresponding branch, which exists below the axis. In fact, it seems to follow, from the very nature of the equation (a), that no curve possessing this property can have both the two points B and C situated in the same branch. These considerations render it necessary, in some measure, to limit the generality of the function φ ; and it may be stated, that no form of φ is admissible, which takes away the double sign placed before the whole of the value of y .

Thus, if we take $\varphi x = ax + \sqrt{c^2 - (ax - x)^2}$, we have

$$y = \frac{ax}{2} \pm \frac{1}{2} \sqrt{c^2 - (ax - x)^2};$$

but

but this does not satisfy the condition of rendering the curve symmetrical relative to its axis.

Let $ax = \sqrt{a^2 - x^2}$, then we have

$$y = \frac{\sqrt{c^2 - a^2 + 2x\sqrt{a^2 - x^2}} \cdot \phi x}{\phi x + \phi \sqrt{a^2 - x^2}};$$

and the following property belongs to all the curves contained in this family :

Any curve BAC of this species being given, Fig. 7., a right line BC given in length may be found such, that if it be any how placed, so that its extremities B and C coincide with any two points of similar branches, but on opposite sides of the axis ; then the sum of the squares described on the abscissæ AE, AD, shall be equal to the square described on a right line which may be found.

If we make $ax = -x$, we have the family of curves expressed by the equation

$$y = \frac{\sqrt{c^2 - 4x^2} \cdot \phi x}{\phi x + \phi(-x)};$$

which possess the following property :

Any one of this class of curves being given, a right line CB may be found, which, if it be placed so that its extremities rest on two opposite branches of the curve (at C and B), and if the ordinate at one of these points be prolonged on the other side the axis, until the part produced (FE) be equal to the abscissa (GH) at the other point, the extremity of the line so produced (E), will be situated in a straight line given by position.

If we make $\phi x = b$, the equation becomes $y = \sqrt{\frac{c^2}{4} - x^2}$,

and the curve is a circle whose radius is $\frac{c}{2}$, the line to be

found

found is equal to twice the radius, and the line given by position passes through the centre, making an angle of $\frac{\pi}{4}$ with the axis.

ADE being any periodic curve of the second order, and AF, AG any two corresponding abscissæ, and ABC being any other curve, whose equation is $y = \psi x$, required the co-ordinates of the point of intersection of its two tangents, at the points B and C.

$$\text{Call } AF = x' \quad AG = x'_1$$

$$BF = y' \quad CG = y'_1;$$

and let x and y be the co-ordinates of any point in either of the tangents, then

$$y = x \frac{dy'}{dx'} - \frac{x' dy'}{dx'} + y'$$

$$y = x \frac{dy'_1}{dx'_1} - \frac{x'_1 dy'_1}{dx'_1} + y'_1$$

are the equations of the tangents; and if we call v and w the co-ordinates of the point of intersection P, we have

$$v = \frac{y' - \frac{x' dy'}{dx'} - y'_1 + \frac{x'_1 dy'_1}{dx'_1}}{\frac{dy'_1}{dx'_1} - \frac{dy'}{dx'}}$$

And

$$w = \frac{\frac{dy'}{dx'} (y' - \frac{x' dy'}{dx'}) - \frac{dy'_1}{dx'_1} (y'_1 - \frac{x'_1 dy'_1}{dx'_1})}{\frac{dy'_1}{dx'_1} - \frac{dy'}{dx'}}$$

Let

Let us now suppose that the point P is always situated in the right line AL, perpendicular to the axis; we have $v = 0$, and since the curve ADE is a periodic one, whose equation is $y = ax$,

$$\psi x - \frac{x a \psi x}{d x} - \psi ax + \frac{ax \cdot d \psi ax}{d ax} = 0$$

Whence $\psi x - \frac{x d \psi x}{d x} = \chi \{x, ax\}$

or $\frac{x d \psi x - \psi x \cdot d x}{x^2} = -\frac{dx}{x^2} \chi \{x, ax\};$

from which $y = \psi x = -x \int \frac{dx}{x^2} \chi \{x, ax\}.$

Now, let $ax = (a^4 - x^4)^{\frac{1}{4}}$, the family of curves are comprehended in the equation

$$y = -x \int \frac{dx}{x^2} \chi \{x, (a^4 - x^4)^{\frac{1}{4}}\};$$

and they possess the following property:

Any of this species of curve being given, if any two abscissæ are taken, the sum of whose fourth powers is equal to the fourth power of a line, which may be found, then the tangent drawn at the extremities of the ordinates corresponding to these abscissæ, will intersect each other in a line given in position. This line is perpendicular to the axis.

If we suppose the point P always to be situated in the axis of the abscissæ, we have $w = 0$, consequently,

$$\frac{dy'}{dx'} \left(y' - \frac{x' dy'}{dx'} \right) - \frac{dy'}{dx'} \left(y' - \frac{x' dy'}{dx'} \right) = 0,$$

D

or

$$\text{or } \frac{d\psi_{ax}}{dax} \left(\psi_x - \frac{x d\psi_x}{dx} \right) = \frac{d\psi_x}{dx} \left(\psi_{ax} - \frac{ax \cdot d\psi_{ax}}{dax} \right),$$

multiplying both sides by $\left(\frac{d\psi_x}{dx} - \frac{d\psi_{ax}}{dax} \right)^{-1}$ we have

$$\psi_x \left(\frac{d\psi_x}{dx} \right)^{-1} - x = \psi_{ax} \left(\frac{d\psi_{ax}}{dax} \right)^{-1} - ax;$$

and since both sides are symmetrical relative to x and ax , we have

$$\psi_x \left(\frac{d\psi_x}{dx} \right)^{-1} - x = \chi(\bar{x}, \overline{ax}),$$

$$\text{or } y \frac{dx}{dy} = x + \chi(\bar{x}, \overline{ax});$$

$$\text{hence, } \frac{dy}{y} = \frac{dx}{x + \chi(\bar{x}, \overline{ax})}.$$

If we suppose $ax = (a^n - x^n)^{\frac{1}{n}}$, the class of curves comprehended in the equation

$$\log y = \int \frac{dx}{x + \chi\left\{x, (a^n - x^n)^{\frac{1}{n}}\right\}}$$

possess the following property :

If we take any two abscissæ AD, AE, the sum of whose nth powers is equal to a given nth power, then the tangents BP, CP, drawn to the curve, at the extremity of the ordinates corresponding to those abscissæ, will always intersect each other on the axis of the abscissæ. Fig. 10.

In the same manner, a class of curves may be found, and ordinates may be drawn, in such a manner, that the tangent

at

at those ordinates shall intersect each other in a line given in position.

In the curve ABC, taking any two ordinates, BE, CD, required its equation when the normals at these points intersect each other on the axis; let x , y , and x_1 , y_1 be the ordinates, then we must have

$$AE + EP = AD + DP,$$

$$\text{or} \quad x + \frac{y \, dy}{dx} = x_1 + \frac{y_1 \, dy_1}{dx_1}$$

if the two points E and D are so chosen, that $x = \alpha x$, and if α is a periodic function of the second order, we have, supposing $y = \psi x$

$$x = \frac{\psi x \cdot d\psi x}{dx} = \alpha x + \frac{\psi \alpha x \cdot d\psi \alpha x}{d\alpha x},$$

$$\text{or} \quad x + \frac{\psi x \cdot d\psi x}{dx} = \chi(\bar{x}, \overline{\alpha x});$$

$$\text{hence} \quad y^2 = -x^2 + 2 \int dx \chi(\bar{x}, \overline{\alpha x}).$$

If we make $\chi(x, \alpha x) = x$, we have

$$y^2 = -x^2 + 2cx + c,$$

which is the equation to a circle, and it is well known, that this curve possesses the property, since all its normals intersect each other in the centre.

This furnishes us with the following local theorems. *Any curve of this family being given, if from any point (P), in the axis of the abscissæ, we draw two lines cutting the same branch of the curve perpendicularly (in B and C), and if we prolong the ordi-*

nate

nate (CD) at one of these points below the axis, until the part below (DF) is equal to the abscissa corresponding to the other point (AE), then the locus of the extremity of the ordinate (F) thus produced, is always a periodic curve of the second order. Fig. 11.

The questions which we have now considered, appear to me sufficient to point out the nature of that connection between the theory of functions and that of curves, which it was my object to establish. The difference between the properties thus brought to light, and those which have been hitherto known, seems to consist chiefly in two points. The first is, that the families of curves to which they relate are larger: this arises from the arbitrary functions necessarily introduced into the solution of functional equations. The other difference is, that the properties discovered relate to many points of the same curve, only connected by some given law. In the first respect, they in some measure approach to the investigations of M. MONGE, in his excellent work *L'Application de l'Analyse a la Geometrie*; whilst, in the second respect, they bear some analogy to the more general properties of curves, deduced from the theory of equations in the *Proprietates Curvarum* of WARING: these resemblances are, however, but superficial. The nature of the questions we have considered requires, by the usual methods of analysis, the application of mixed differences; and, in most of the few instances in which any such problems have been proposed, they have been attempted by that method.

Devonshire Street, Portland Place, }
July 1. 1818. }

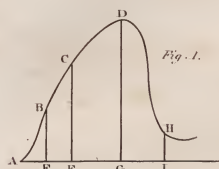


Fig. 1.

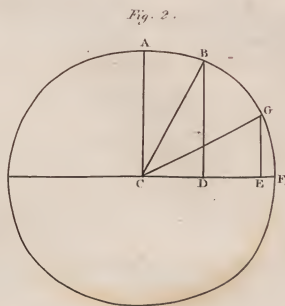


Fig. 2.

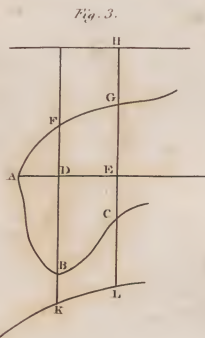


Fig. 3.

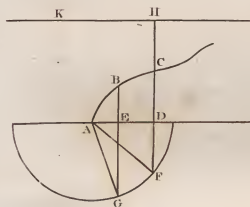


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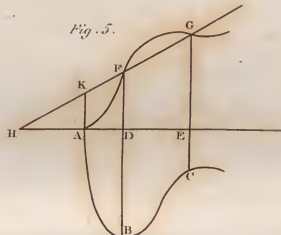


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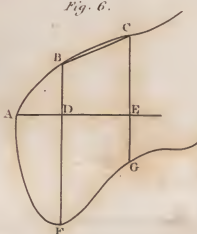


Fig. 7.

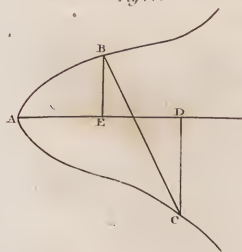


Fig. 8.

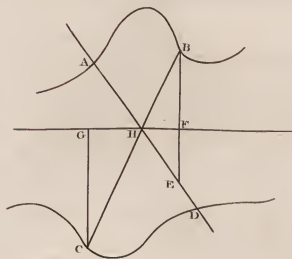


Fig. 9.

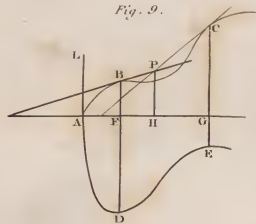


Fig. 10.

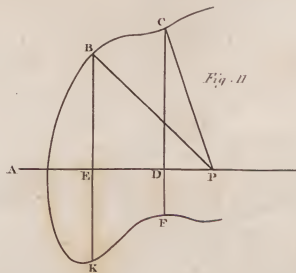
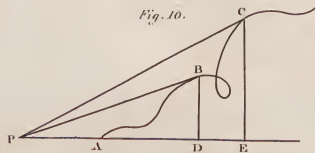


Fig. 11.

